# Multi-factor Foldovers by Using Geometrical Designs

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*Using fractional factorial designs often leads to great economy and efficiency for sequential experimentation, in which a foldover method plays an important role. In this article, three cases of single-factor, multi-factor, and complete foldover designs for sequential experimentations were presented using geometrical designs proposed by Plackett & Burman (1946). A design matrix with complete alias structure is visually appealing for analysis purpose. The results show that geometrical designs with nice alias structure are useful tools for sequential experimentations.*

**Field of Research:** Management, Operation Management

## 1. Introduction

Two-level fractional factorial experiments have been used extensively in many areas such as quality engineering, business management, conjoint analysis, agriculture, pharmaceutical chemistry, etc.. Resolution and minimum aberration are two important criteria for choosing a good fractional factorial design, in which the wordlength patterns (WLP's) are required for comparisons. A defining relation is a word of letters (factors) denoted by 1, 2, 3… (or A, B, C…), and the number of letters in a word is called as its wordlength. A $2^{k-p}$ fractional factorial design d with n factors is uniquely determined by p independent defining relations (words) which generate the defining contrast subgroup. Then, the wordlength pattern of a design d is defined as the vector WLP(d)=[$A_1$(d), $A_2$(d), …, $A_k$(d)], where $A_j$(d) is the number of length-j words in the defining contrast subgroup. The resolution of a design is the smallest r satisfying $A_r \geqq 1$, however, two designs having the same resolution may have different wordlength patterns (WLP's). Based on a WLP, Fries & Hunter (1980) proposed the minimum aberration criterion to further discriminate $2^{n-p}$ fractional factorial designs. Let r be the smallest integer such that $A_r(d_1) \neq A_r(d_2)$ for any two designs $d_1$ and $d_2$, $d_1$ has less aberration then $d_2$ if $A_r(d_1) < A_r(d_2)$. Then, a design d* has minimum aberration if no design has less aberration than d*.

Using fractional factorial designs often leads to great economy and efficiency in experimentation, particularly if the runs can be made sequentially. A foldover method plays an important role in sequential experimentations (Box & Wilson, 1951; Box & Hunter, 1961; Box, Hunter, & Hunter 1978). Montgomery (1991) discussed sequential assembly of fractions to separate effects, which are computed by making changes of sign on factors reversed in the alias structure of the original fraction. Montgomery & Runger (1996)

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defined a foldover of a $2^{k-p}$ design with any resolution as the procedure of adding a new $2^{k-p}$ fraction in which signs are reversed on one or more factors, hence, there are single-factor, multi-factor, and complete (full) foldover designs. They used two simple foldover rules to developed foldover designs by using sing-reversal method. Li & Lin (2003) obtained the optimal foldover plans for two-level fractional factorial designs with 16 and 32 runs and tabulated the results for practical use. Mee & Xiao (2008) found the optimal foldovers and semifolding for minimum aberration even fractional factorial designs for sequential experimentations using foldover methods and exhaustively search method, in which the optimal number of factors reversed is searched.

Tsai (1999) showed that geometrical designs proposed by Plackett & Burman (1946) with nice alias structures are useful tools for planning two-level fractional factorial experiments, and they will be adopted here to facilitate the development of sequential experimentations. In this article, the alias structure of a combined design is presented by using geometrical designs, which is visually appealing and can help users understand its application easily. Three cases of single-factor foldover, two-factor foldover, and complete (full) foldover designs will be discussed below.

## 2. Geometrical Designs

Plackett & Burman (1946) first proposed geometrical designs by using a doubling method: if $G_n$ is orthogonal, $\mathbf{G_{2n}} = \begin{bmatrix} \mathbf{G_n} & \mathbf{G_n} \\ \mathbf{G_n} & -\mathbf{G_n} \end{bmatrix}$ is also orthogonal and has double the order of $G_n$. Note that $G_{2n}$ can be expressed as $G_{2n}$=[$LG_n$, $RG_n$] where the left-half part $LG_{2n}$=[$G_n$, $G_n$]', the right-half part $RG_{2n}$=[$G_n$, -$G_n$]', and $RG_{2n}$ is called "foldover" by Box & Wilson (1951). By applying Kronecker product $\otimes$ successively, a higher order geometrical design can be obtained easily from. For example, starting from $\mathbf{G_2} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, then we can have $\mathbf{G_{2^r}} = \mathbf{G_2} \otimes \mathbf{G_2} \otimes ... \otimes \mathbf{G_2} \otimes \mathbf{G_2}$ by applying successive doubling method r-1 times. In general, $\mathbf{G_{2n}} = \mathbf{G_n} \otimes \mathbf{G_2} = \begin{bmatrix} \mathbf{G_n} & \mathbf{G_n} \\ \mathbf{G_n} & -\mathbf{G_n} \end{bmatrix}$, n=2,4,…$2^r$. The design matrices of $G_2$, $G_4$, and $G_8$ are listed below:

$$\mathbf{G_2} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \mathbf{G_4} = \begin{bmatrix} \mathbf{G_2} & \mathbf{G_2} \\ \mathbf{G_2} & -\mathbf{G_2} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}, \text{ and}$$

$$\mathbf{G_8} = \mathbf{G_4} \otimes \mathbf{G_2} = \begin{bmatrix} \mathbf{G_4} & \mathbf{G_4} \\ \mathbf{G_4} & -\mathbf{G_4} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix}.$$

Similarly, all geometrical designs with higher order such as $G_{16}$, $G_{32}$, $G_{64}$, etc. could be obtained easily by hand writing, or by using spread sheet such as "Excel" with functions of "copy" and "replace" without any algebraic computation.

## 3. Alias Structure of $G_n$

In this section, a complete alias structure, in which all multi-factor interactions are shown under the same column for each alias set, is proposed by using geometrical design matrix for analysis purpose. Tsai (1999) showed that a geometrical design $G_{2n}$ has a nice alias structure, and the two-factor interactions have the following rules:

R1: The interaction column of any two given columns both in $LG_{2n}$ remains in $LG_{2n}$.

R2: The interaction column of any two given columns both in $RG_{2n}$ remains in $LG_{2n}$.

R3: The interaction column of any two given columns from different half parts respectively remains $RG_{2n}$.

In general, let $a \oplus b$ denote the interaction of two columns a and b, where $\oplus$ is addition of module 2 for binary numbers, then $a \oplus b \in LG_{2n}$, if $a,b \in LG_{2n}$ or $RG_{2n}$; $a \oplus b \in RG_{2n}$, if $a \in LG_{2n}$ and $b \in RG_{2n}$. Note that factors and columns are used interchangeably throughout this article. Furthermore, the following recursive equations for consecutive design sizes can also be found in Tsai (1999). Let $T_n(i,j)$ denote the interaction column of two given column i and j, $0 \leqq i,j<n$ in a $G_n$ design, then

F1: $T_{2n}(i,j)=T_n(i,j)$, $0 \leqq i,j<n$. (R1)

F2: $T_{2n}(i,j)=T_n(i-n,j-n)$, $n \leqq i,j<2n$. (R2)

F3: $T_{2n}(i,j)=T_n(i,j-n)+n$, $0 \leqq i<n \leqq j<2n$. (R3)

Another easy way is to compute an interaction of multi-factor by $\oplus$, addition of module 2, on the binary column numbers of a geometrical design directly, for examples,

(1) $T_{16}(6,9)=T_{16}([0110] \oplus [1001])=[1111]=15$.

(2) $T_8(1,2,4)=T_8([001] \oplus [010] \oplus [001])=[111]=7$.

With all these useful tools, the alias structure of a geometrical design can be obtained easily. Table 1 shows the design matrix and the complete alias structure of a geometrical design $G_8(2^{7-4})$ which has k=7 factors and n=8 runs. Some interesting characteristics and special designs are discussed below:

(1) A geometrical design $G_8=LG_8 \cup RG_8$, where $LG_8=\{0,1,2,3\}$ and $RG_8=\{4,5,6,7\}$, has p=4 generated columns (factors), say 4,5,6, and 7; $2^4-1=15$ words in the defining relation, say {123, 145, 167,…, 1234567} in the identity column I; and its WLP=[0,0,7,7,0,0,1]

(2) The alias structure and the design matrix of a geometrical design are presented together in a spreadsheet which can facilitate the computation of all effects.

(3) Two-factor interactions {23,13,12} are in $LG_8$ which verify R1; {45,46,47, 67,57,56} are in $LG_8$ which verify R2; {15,14,17,16,26,27,24,25,37,36,35,34} are in $RG_8$ which verify R3. The higher order interactions can be verified by the similar manners sequentially.

(4) By F2, $T_8(5,7)=T_4(1,3)=2$ shows that the interaction of columns 5 and 7 can be computed as columns 5-4=1 and 7-4=3 which is confounded with column 2.

(5) By F3, $T_8(1,7)=T_4(1,7-4)+4=2+4=6$ shows that the interaction of columns 1

and 7 can be computed as columns 1 and 7-4=3 which is confounded with column 2+4=6.

(6) If all factors are assigned to $RG_{2n}$, then it is an even design. For example, $d_1$={4,5,6,7}, then the only word is I=4567, even-factor interactions {45,46,47,67,57,56} are in $LG_8$ while odd-factor interactions {567,467,457,456} are in $RG_8$.

(7) If all factors are assigned to $LG_{2n}$, then it is a duplicated design since $LG_{2n}=[G_n, G_n]'$ in which $G_n$ is duplicated. For example, say $d_2$={1,2,3} in $G_8$, or $d_3$={1,2,3,4,5,6,7} in $G_{16}$, etc..

Table 1: The design matrix and the complete alias structure of a geometrical design $G_8(2^{7-4})$.

| **I** | **A** | **B** | **C** | D | E | F | G |
|---|---|---|---|---|---|---|---|
| **0** | **1** | **2** | **3** | 4 | 5 | 6 | 7 |
| **1** | **1** | **1** | **1** | 1 | 1 | 1 | 1 |
| **1** | **-1** | **1** | **-1** | 1 | -1 | 1 | -1 |
| **1** | **1** | **-1** | **-1** | 1 | 1 | -1 | -1 |
| **1** | **-1** | **-1** | **1** | 1 | -1 | -1 | 1 |
| **1** | **1** | **1** | **1** | -1 | -1 | -1 | -1 |
| **1** | **-1** | **1** | **-1** | -1 | 1 | -1 | 1 |
| **1** | **1** | **-1** | **-1** | -1 | -1 | 1 | 1 |
| **1** | **-1** | **-1** | **1** | -1 | 1 | 1 | -1 |
| **123** | **23** | **13** | **12** | 15 | 14 | 17 | 16 |
| **145** | **45** | **46** | **47** | 26 | 27 | 24 | 25 |
| **167** | **67** | **57** | **56** | 37 | 36 | 35 | 34 |
| **246** | **247** | **147** | **146** | 127 | 126 | 125 | 124 |
| **257** | **256** | **156** | **157** | 136 | 137 | 134 | 135 |
| **347** | **346** | **345** | **245** | 235 | 234 | 237 | 236 |
| **356** | **357** | **367** | **267** | 567 | 467 | 457 | 456 |
| **1247** | **1246** | **1245** | **1345** | 1234 | 1235 | 1236 | 1237 |
| **1256** | **1257** | **1267** | **1367** | 1467 | 1567 | 1456 | 1457 |
| **1346** | **1347** | **2347** | **2346** | 2457 | 2456 | 2567 | 2467 |
| **1357** | **1356** | **2356** | **2357** | 3456 | 3457 | 3467 | 3567 |
| **2345** | **12345** | **12346** | **12347** | 12456 | 12457 | 12467 | 12567 |
| **2367** | **12367** | **12357** | **12356** | 13457 | 13456 | 13567 | 13467 |
| **4567** | **14567** | **24567** | **34567** | 23467 | 23567 | 23456 | 23457 |
| **1234567** | **234567** | **134567** | **124567** | 123567 | 123467 | 123457 | 123456 |

## 4. Sequential experimentations by geometrical designs

Using fractional factorial designs often leads to great economy and efficiency in experimentation, particularly if the runs can be made sequentially. A two-level fractional factorial design is used first, i.e. $2^{k-p}$ design, where k is the number of factors, $2^p$ is the number of fractions; a second fraction is added if it is necessary, say another $2^{k-p}$ design; continue until it is satisfied. Three cases of single-factor, two-factor, and complete foldover designs will be discussed below.

### Single-factor foldover

A second fraction with the signs of a single factor reversed is add to a resolution III fractional factorial design or higher, and then the combined design can estimate the main effect of that factor and its two-factor interactions. Table 2 shows that the design matrix and the complete alias structure of the combined $G_{16}(2^{7-3})$ with factor G reversed in the original $G_8(2^{7-4})$, and some interesting facts are observed as follows.

(1) It is interesting to note that the factor G is at column 7 in the original $G_8$, after reversed, it is moved to column 15 (G'), by adding the original run size 8, in the combined $G_{16}$, say, the original design is $d_4$={1,2,3,4,5,6,7}

whereas the combined design is $d_5$={1,2,3,4,5,6,15}.

(2) There are $2^4$-1=15 words in the original $G_8$ (see Table 1), since column 7 is changed to column 15 after factor G is reversed, eight words with 7 in the original design are no longer words in the combined design and are confounded with column 8 by using (F3), for example, I=167 is a word in the $G_8$ while 8=16<u>15</u> is not a word in the combined $G_{16}$.

(3) The seven words without 7 (G) such as {123, 145, 246, 356, 1256, 1346, 2345} in the original $G_8$ (see Table 1) will remain as words since they are not affected by reversing.

(4) The original wordlength pattern is WLP=[0,0,7,7,0,0,1] whereas the new one is WLP=[0,0,4,3,0,0,0].

(5) The two-factor interaction of factor G with other factors such as {6<u>15</u>, 5<u>15</u>, 4<u>15</u>,3<u>15</u>,2<u>15</u>,1<u>15</u>} are in columns 9-14, which are columns 1-6 by adding 8 respectively, and are not confounded with any main effects (columns 1-6) or two-factor interactions, namely, they are de-aliased or clear. On the contrast, all factors A-F without reversed are confounded with their own two-factor interactions in columns 1-6.

(6) Note that all effects can be computed directly on the design matrix of a geometrical design by using spreadsheet easily.

Table 2: The design matrix and the complete alias structure of combined $G_{16}$ with factor G reversed.

| 0 (I) | 1 (A) | 2 (B) | 3 (C) | 4 (D) | 5 (E) | 6 (F) | 7 (G) |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | -1 | 1 | -1 | 1 | -1 | 1 | -1 |
| 1 | 1 | -1 | -1 | 1 | 1 | -1 | -1 |
| 1 | -1 | -1 | 1 | 1 | -1 | -1 | 1 |
| 1 | 1 | 1 | 1 | -1 | -1 | -1 | -1 |
| 1 | -1 | 1 | -1 | -1 | 1 | -1 | 1 |
| 1 | 1 | -1 | -1 | -1 | -1 | 1 | 1 |
| 1 | -1 | -1 | 1 | -1 | 1 | 1 | -1 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | -1 | 1 | -1 | 1 | -1 | 1 | -1 |
| 1 | 1 | -1 | -1 | 1 | 1 | -1 | -1 |
| 1 | -1 | -1 | 1 | 1 | -1 | -1 | 1 |
| 1 | 1 | 1 | 1 | -1 | -1 | -1 | -1 |
| 1 | -1 | 1 | -1 | -1 | 1 | -1 | 1 |
| 1 | 1 | -1 | -1 | -1 | -1 | 1 | 1 |
| 1 | -1 | -1 | 1 | -1 | 1 | 1 | -1 |
| I | A | B | C | D | E | F | 1,6 |
| 1,2,3 | 2,3 | 1,3 | 1,2 | 1,5 | 1,4 | 2,4 | 2,5 |
| 1,4,5 | 4,5 | 4,6 | 5,6 | 2,6 | 3,6 | 3,5 | 3,4 |
| 2,4,6 | 2,5,6 | 1,5,6 | 1,4,6 | 1,3,6 | 1,2,6 | 1,2,5 | 1,2,4 |
| 3,5,6 | 3,4,6 | 3,4,5 | 2,4,5 | 2,3,5 | 2,3,4 | 1,3,4 | 1,3,5 |
| 1,2,5,6 | 1,2,4,6 | 1,2,4,5 | 1,3,4,5 | 1,2,3,4 | 1,2,3,5 | 1,2,3,6 | 2,3,6 |
| 1,3,4,6 | 1,3,5,6 | 2,3,5,6 | 2,3,4,6 | 3,4,5,6 | 2,4,5,6 | 1,4,5,6 | 4,5,6 |
| 2,3,4,5 | 1,2,3,4,5 | 1,2,3,4,6 | 1,2,3,5,6 | 1,2,4,5,6 | 1,3,4,5,6 | 2,3,4,5,6 | 1,2,3,4,5,6 |

| 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | -1 | 1 | -1 | 1 | -1 | 1 | -1 |
| 1 | 1 | -1 | -1 | 1 | 1 | -1 | -1 |
| 1 | -1 | -1 | 1 | 1 | -1 | -1 | 1 |
| 1 | 1 | 1 | 1 | -1 | -1 | -1 | -1 |
| 1 | -1 | 1 | -1 | -1 | 1 | -1 | 1 |
| 1 | 1 | -1 | -1 | -1 | -1 | 1 | 1 |
| 1 | -1 | -1 | 1 | -1 | 1 | 1 | -1 |
| -1 | -1 | -1 | -1 | -1 | -1 | -1 | **-1** |
| -1 | 1 | -1 | 1 | -1 | 1 | -1 | **1** |
| -1 | -1 | 1 | 1 | -1 | -1 | 1 | **1** |
| -1 | 1 | 1 | -1 | -1 | 1 | 1 | **-1** |
| -1 | -1 | -1 | -1 | 1 | 1 | 1 | **1** |
| -1 | 1 | -1 | 1 | 1 | -1 | 1 | **-1** |
| -1 | -1 | 1 | 1 | 1 | 1 | -1 | **-1** |
| -1 | 1 | 1 | -1 | 1 | -1 | -1 | **1** |
| **1,6,15** | **6,15** | **5,15** | **4,15** | **3,15** | **2,15** | **1,15** | **G'** |
| **2,5,15** | **2,4,15** | **1,4,15** | **1,5,15** | **1,2,15** | **1,3,15** | **2,3,15** | **1,2,3,15** |
| **3,4,15** | **3,5,15** | **3,6,15** | **2,6,15** | **5,6,15** | **4,6,15** | **4,5,15** | **1,4,5,15** |
| **1,2,4,15** | **1,2,5,15** | **1,2,6,15** | **1,3,6,15** | **1,4,6,15** | **1,5,6,15** | **2,5,6,15** | **2,4,6,15** |
| **1,3,5,15** | **1,3,4,15** | **2,3,4,15** | **2,3,5,15** | **2,4,5,15** | **3,4,5,15** | **3,4,6,15** | **3,5,6,15** |
| **2,3,6,15** | **1,2,3,6,15** | **1,2,3,5,15** | **1,2,3,4,15** | **1,3,4,5,15** | **1,2,4,5,15** | **1,2,4,6,15** | **1,2,5,6,15** |
| **4,5,6,15** | **1,4,5,6,15** | **2,4,5,6,15** | **3,4,5,6,15** | **2,3,4,6,15** | **2,3,5,6,15** | **1,3,5,6,15** | **1,3,4,6,15** |
| **1,2,3,4,5,6,15** | **2,3,4,5,6,15** | **1,3,4,5,6,15** | **1,2,4,5,6,15** | **1,2,3,5,6,15** | **1,2,3,4,6,15** | **1,2,3,4,5,15** | **2,3,4,5,15** |

## Two-factor foldover

A second fraction with the signs of two or more factors reversed is add to a resolution III fractional factorial design or higher, and then the combined design can estimate the main effects of those factors and their two-factor interactions. Table 3 shows that the complete alias structure of the combined $G_{16}(2^{7-3})$ with factors F & G reversed in the original $G_8(2^{7-4})$, and some interesting facts are observed as follows.

(1) The design matrix of the combined $G_{16}$ is omitted since it is identical to that in Table 2.

(2) It is interesting to note that the factors F & G are at columns 6 & 7 in the original $G_8$, after reversed, they are moved to column 14 & 15 (F' & G'), by adding the original run size 8, in the combined $G_{16}$, say, the original design is $d_4$={1,2,3,4,5,6,7} whereas the combined design is $d_6$={1,2,3,4,5,14,15}.

(3) There are $2^4$-1=15 words in the original $G_8$ (see Table 1), since columns 6 & 7 are changed to columns 14 & 15 after reversing, eight words with either column 14 or 15 are no longer words in the combined design and are confounded with column 8 by using (F3), for example, 8=2414 or 8=2515 is not a word in the combined $G_{16}$.

(4) The seven words with either both or without columns 14 & 15 such as {123, 145, 114 15, 2345, 2314 15, 4514 15, 1234514 15} are words by using (F2).

(5) The original wordlength pattern is WLP=[0,0,7,7,0,0,1] whereas the new one is WLP=[0,0,3,3,0,0,1].

(6) In this case, all two-factor interactions are confounded with either main effects or other two-factor interactions, namely, none of them are clear. Note that the interaction of columns 14 & 15 is confounded with column 1 by using (F2).

Table 3: The complete alias structure of combined $G_{16}$ with factors F & G reversed.

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| **I** | **A** | **B** | **C** | **D** | **E** | 2,4 | 2,5 |
| **1,2,3** | 2,3 | 1,3 | 1,2 | 1,5 | 1,4 | 3,5 | 3,4 |
| **1,4,5** | 4,5 | 3,4,5 | 2,4,5 | 2,3,5 | 2,3,4 | 1,2,5 | 1,2,4 |
| **1,14,15** | 14,15 | 3,14,15 | 2,14,15 | 5,14,15 | 4,14,15 | 1,3,4 | 1,3,5 |
| **2,3,4,5** | 1,2,3,14,15 | 1,2,4,5 | 1,3,4,5 | 1,2,3,4 | 1,2,3,5 | 2,5,14,15 | 2,4,14,15 |
| **2,3,14,15** | 1,4,5,14,15 | 1,2,14,15 | 1,3,14,15 | 1,4,14,15 | 1,5,14,15 | 3,4,14,15 | 3,5,14,15 |
| **4,5,14,15** | 1,2,3,4,5 | 2,4,5,14,15 | 3,4,5,14,15 | 2,3,4,14,15 | 2,3,5,14,15 | 1,2,4,14,15 | 1,2,5,14,15 |
| **1,2,3,4,5,14,15** | 2,3,4,5,14,15 | 1,3,4,5,14,15 | 1,2,4,5,14,15 | 1,2,3,5,14,15 | 1,2,3,4,14,15 | 1,3,5,14,15 | 1,3,4,14,15 |

| 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|
| **2,4,14** | **2,4,15** | **4,14** | **4,15** | **2,14** | **2,15** | **F'** | **G'** |
| **2,5,15** | **2,5,14** | **5,15** | **5,14** | **3,15** | **3,14** | **1,15** | **1,14** |
| **3,4,15** | **3,4,14** | **1,4,15** | **1,4,14** | **1,2,15** | **1,2,14** | **2,3,15** | **2,3,14** |
| **3,5,14** | **3,5,15** | **1,5,14** | **1,5,15** | **1,3,14** | **1,3,15** | **4,5,15** | **4,5,14** |
| **1,2,4,15** | **1,2,4,14** | **2,3,4,15** | **2,3,4,14** | **2,4,5,15** | **2,4,5,14** | **1,2,3,14** | **1,2,3,15** |
| **1,2,5,14** | **1,2,5,15** | **2,3,5,14** | **2,3,5,15** | **3,4,5,14** | **3,4,5,15** | **1,4,5,14** | **1,4,5,15** |
| **1,3,4,14** | **1,3,4,15** | **1,2,3,4,14** | **1,2,3,4,15** | **1,2,4,5,14** | **1,2,4,5,15** | **2,3,4,5,14** | **2,3,4,5,15** |
| **1,3,5,15** | **1,3,5,14** | **1,2,3,5,15** | **1,2,3,5,14** | **1,3,4,5,15** | **1,3,4,5,14** | **1,2,3,4,5,15** | **1,2,3,4,5,14** |

**Complete Foldover**

A second fraction with the signs of all factors reversed is added to a resolution III fractional factorial design, and then the combined design can estimate all the main effects clear of any two-factor interactions**.** Table 4 shows that the alias structure of the combined $G_{16}(2^{7-3})$ with all factors reversed in the original $G_8(2^{7-4})$, and some interesting facts are observed as follows.

(1) The combined design is $d_7=\{9,\underline{10},\underline{11},\underline{12},\underline{13},\underline{14},\underline{15}\}$.

(2) All the odd words in the original design are confounded with column 8 and no longer words in the combined design, for example, a word of I=123 in $G_8(2^{7-4})$ becomes $8=9\underline{10}\ \underline{11}$ which is not a word in $G_{16}(2^{7-3})$.

(3) In this case, only even words in the original design remain as words in the combined design, for example, a word of I=1247 becomes $I=9\underline{10}\ \underline{12}\ \underline{15}$.

(4) The combined wordlength pattern is WLP=[0,0,0,7,0,0,0] which has even resolution IV and minimum aberration under reversing method.

(5) Anyone main effect is confounded with four three-factor interactions and three five-factor interactions; each of seven alia sets contains three two-factor interactions.

Table 4: The complete alias structure of a foldover combined design $G_{16}(2^{7-3})$.

| **0** | **1 (A)** | **2 (B)** | **3 (C)** | **4 (D)** | **5 (E)** | **6 (F)** | **7 (G)** |
|---|---|---|---|---|---|---|---|
| **I** | 10,11 | 9,11 | 9,10 | 9,13 | 9,12 | 9,15 | 9,14 |
| **9,10,12,15** | 12,13 | 12,14 | 12,15 | 10,14 | 10,15 | 10,12 | 10,13 |
| **9,10,13,14** | 14,15 | 13,15 | 13,14 | 11,15 | 11,14 | 11,13 | 11,12 |
| **9,11,12,14** | 9,10,12,14 | 9,10,12,13 | 9,11,12,13 | 9,10,11,12 | 9,10,11,13 | 9,10,11,14 | 9,10,11,15 |
| **9,11,13,15** | 9,10,13,15 | 9,10,14,15 | 9,11,14,15 | 9,12,14,15 | 9,13,14,15 | 9,12,13,14 | 9,12,13,15 |
| **10,11,12,13** | 9,11,12,15 | 10,11,12,15 | 10,11,12,14 | 10,12,13,15 | 10,12,13,14 | 10,13,14,15 | 10,12,14,15 |
| **10,11,14,15** | 9,11,13,14 | 10,11,13,14 | 1011,13,15 | 11,12,13,14 | 11,12,13,15 | 11,12,14,15 | 11,13,14,15 |
| **12,13,14,15** | 10,11,12,13, 14,15 | 9,11,12,13, 14,15 | 9,10,12,13, 14,15 | 9,10,11,13, 14,15 | 9,10,11,12, 14,15 | 9,10,11,12, 13,15 | 9,10,11,12, 13,14 |
| **8** | **9** | **10** | **11** | **12** | **13** | **14** | **15** |
| **9,10,11** | **A'** | **B'** | **C'** | **D'** | **E'** | **F'** | **G'** |
| **9,12,13** | 10,12,15 | 9,12,15 | 9,12,14 | 9,10,15 | 9,10,14 | 9,10,13 | 9,10,12 |
| **9,14,15** | 10,13,14 | 9,13,14 | 9,13,15 | 9,11,14 | 9,11,15 | 9,11,12 | 9,11,13 |
| **10,12,14** | 11,12,14 | 11,12,13 | 10,12,13 | 10,11,13 | 10,11,12 | 10,11,15 | 10,11,14 |
| **10,13,15** | 11,13,15 | 11,14,15 | 10,14,15 | 13,14,15 | 12,14,15 | 12,13,15 | 12,13,14 |
| **11,12,15** | 9,10,11,12,13 | 9,10,11,12,14 | 9,10,11,12,15 | 9,10,12,13,14 | 9,10,12,13,15 | 9,10,12,14,15 | 9,10,13,14,15 |
| **11,13,14** | 9,10,11,14,15 | 9,10,11,13,15 | 9,10,11,13,14 | 9,11,12,13,15 | 9,11,12,13,14 | 9,11,13,14,15 | 9,11,12,14,15 |
| **9,10,11,12, 13,14,15** | 9,12,13,14,15 | 10,12,13,14,15 | 11,12,13,14,15 | 10,11,12,14,15 | 10,11,13,14,15 | 10,11,12,13,14 | 10,11,12,13,15 |

## 5. Conclusion

Three cases of single-factor, multi-factor, and complete foldover designs for sequential experimentations were presented by using geometrical designs proposed by Plackett & Burman (1946), and their alias structures as well as some interesting facts were discussed in details. Using geometrical designs to present sequential experimentations has the following advantages: (1) it is visually appealing to see the complete alias structure physically; (2) all effects can be computed on its design matrix by using spreadsheet easily; (3) it can help users understand its application easily. The results show that geometrical designs with nice alias structure are useful tools for sequential experimentations. Due to their nice characteristics, geometrical designs have theoretical interests and practical attractiveness.

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